

Some Ricci Flat (pseudo-) Riemannian Geometries

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Abstract

We define a class of two dimensional surfaces conformally related to minimal surfaces in flat three dimensional geometries. By the utility of the metrics of such surfaces we give a construction of the metrics of $2N$ dimensional Ricci flat (pseudo-) Riemannian geometries.

1 Introduction

Let (S, g) denote a two dimensional geometry where S is a surface in a three dimensional flat manifold M_3 and g is a (pseudo-) Riemannian metric on S with a non vanishing determinant, $\det(g)$. Furthermore we assume that g satisfies the following conditions

$$\partial_\mu (g^{\mu\nu} g^{-1} \partial_\nu g) = 0, \tag{1}$$

$$R + \frac{1}{4} \text{tr}[g^{\mu\nu} \partial_\mu g^{-1} \partial_\nu g] = 0. \tag{2}$$

where R is the Ricci scalar (Gaussian curvature) of S (please see the next section for our conventions). We shall see in the following sections that some surfaces which are conformally related to minimal surfaces satisfy the above conditions.

The importance of such surfaces arises when we are interested in even dimensional Ricci flat geometries. By the utility the metric g of these surfaces we shall give a construction (without solving any further differential equations) of the metric of a $2N$ dimensional Ricci flat (pseudo-) Riemannian geometries.

Ricci flat geometries are important not only in differential geometry and general relativity but also in gravitational instantons and in brane solutions of string theory [1].

2 Conformally related minimal surfaces

Let ϕ be a differentiable function of x^1 and x^2 and S_0 be the surface in a three dimensional manifold M_3 (not necessarily Euclidean) with a pseudo-Riemannian metric g_3 defined through $ds^2 = g_{0\mu\nu} dx^\mu dx^\nu + \epsilon (dx^3)^2$, where $\mu, \nu = 1, 2$, $\epsilon = \pm 1$ and g_0 is a constant, invertible, symmetric 2×2 matrix. In this work we assume Einstein summation convention, i.e., the repeated indices are summed up. S_0 is given as the graph of the function ϕ , i.e., $S_0 = \{(x^1, x^2, x^3) \in M_3 | x^3 = \phi(x^1, x^2)\}$. Then the metric on S_0 is given by

$$h_{\mu\nu} = g_{0\mu\nu} + \epsilon \phi_{,\mu} \phi_{,\nu}. \quad (3)$$

Since $\det h = (\det g_0) \rho$ where

$$\rho = 1 + \epsilon g_0^{\mu\nu} \phi_{,\mu} \phi_{,\nu} \quad (4)$$

then h is everywhere (except at those points where $\rho = 0$) invertible. Its inverse is given by

$$h^{\mu\nu} = g_0^{\mu\nu} - \frac{\epsilon}{\rho} \phi^\mu \phi^\nu \quad (5)$$

where $g_0^{\mu\nu}$ are the components of the inverse matrix g_0^{-1} of g_0 . Here the indices are lowered and raised by the metric g_0 and its inverse g_0^{-1} respectively. For instance, $\phi^\mu_{,\nu} = g_0^{\mu\alpha} \phi_{,\alpha\nu}$.

The Ricci tensor corresponding to the metric in (3) is given by

$$r_{\mu\nu} = \frac{\epsilon}{\rho} (\nabla^2 \phi) \phi_{,\mu\nu} - \frac{\epsilon}{\rho} \phi_{,\mu}{}^\alpha \phi_{,\nu\alpha} + \frac{1}{4\rho^2} \rho_{,\mu} \rho_{,\nu}, \quad (6)$$

where

$$\nabla^2 \phi = h^{\mu\nu} \phi_{,\mu\nu} = g_0^{\mu\nu} \phi_{,\mu\nu} - \frac{1}{2\rho} \phi^\alpha \rho_{,\alpha} \quad (7)$$

The Ricci scalar or the Gaussian curvature K and the minimal curvature H are obtained as

$$K = \frac{\epsilon}{\rho^2} [(\phi^\alpha{}_\alpha)^2 - \phi^{\alpha\beta} \phi_{,\alpha\beta}] \quad (8)$$

$$H = \frac{1}{\sqrt{\rho}} h^{\mu\nu} \phi_{,\mu\nu}, \quad (9)$$

The following proposition is valid only for the case of two dimensional geometries.

Proposition 1. $\phi_{,\alpha\mu} \phi_{,\beta\gamma} - \phi_{,\alpha\beta} \phi_{,\mu\gamma} = -\lambda_0 (g_{0\alpha\mu} g_{0\beta\gamma} - g_{0\alpha\beta} g_{0\gamma\mu})$

where

$$\lambda_0 = \frac{1}{2} [\phi^{\alpha\beta} \phi_{\alpha\beta} - (\phi^\alpha{}_\alpha)^2]. \quad (10)$$

The following corollaries will be very useful in this section

Corollary 1. $\phi_\mu^\alpha \phi_{,\alpha\nu} - \phi_\alpha^\alpha \phi_{,\mu\nu} = \lambda_0 g_{0\mu\nu}$

Corollary 2. $r_{\alpha\beta} = \frac{K}{2} h_{\alpha\beta}, \quad \lambda_0 = -\frac{\epsilon}{2} \rho^2 K$

For the minimal surfaces we have $H = 0$ and the following result

Proposition 2. If $H = 0$ then ([2])

$$\partial_\alpha [\sqrt{\rho} h^{\alpha\beta} \partial_\beta \phi] = 0, \quad (11)$$

$$\partial_\alpha (\sqrt{\rho} h^{\alpha\beta}) = 0. \quad (12)$$

We now define surfaces which are locally conformal to minimal surfaces. Let S be such a surface, i.e., locally conformal to S_0 . Then the metric on S is given by

$$g_{\alpha\beta} = \frac{1}{\sqrt{\rho}} h_{\alpha\beta}. \quad (13)$$

It is clear that $\det g = \det g_0 \neq 0$. In the sequel we shall assume that the surface S_0 is minimal and hence the metric defined on it satisfies all the equivalent conditions in proposition 2. The corresponding Ricci tensor of g is given as

$$R_{\alpha\beta} = r_{\alpha\beta} - (\nabla_g^2 \psi_0) g_{\alpha\beta}, \quad (14)$$

where $\psi_0 = -\frac{1}{4} \log(\rho)$ and ∇_g^2 is the Laplace-Beltrami operator with respect to the metric g . Using the above results we have

Proposition 3. The following are some identities related to the conformally related surface S .

$$R = -\frac{1}{4} g^{\alpha\beta} \text{tr}[\partial_\alpha g^{-1} \partial_\beta g] \quad (15)$$

$$R_{\alpha\beta} = -\frac{\rho - 1}{2} r_{\alpha\beta}, \quad (16)$$

$$R = \sqrt{\rho} r - 2\nabla_h^2 \psi_0. \quad (17)$$

Here g is the 2×2 matrix of $g_{\alpha\beta}$ and g^{-1} is its inverse. The operation tr is the standard trace operation for matrices.

Let $v_\alpha = (1, 0)$, $v'_\alpha = (0, 1)$ and $u^\alpha = (1, 0)$, $u'^\alpha = (0, 1)$. We now define some functions over S .

$$\xi_1 = g^{\alpha\beta} v_\alpha v_\beta, \quad \xi_2 = g^{\alpha\beta} v'_\alpha v'_\beta, \quad (18)$$

$$w_1 = \sqrt{\rho} g_{\alpha\beta} u^\alpha u^\beta, \quad w_2 = \sqrt{\rho} g_{\alpha\beta} u'^\alpha u'^\beta. \quad (19)$$

It is now easy to prove

Proposition 4.

$$\nabla_g^2 \zeta - a_0 R = -a_0 \sqrt{\rho} K, \quad (20)$$

$$\nabla_g^2 \psi_1 - (a_1 + a_2) R = 0 \quad (21)$$

$$\nabla_g^2 \psi_2 - 2(b_1 + b_2) R = -(b_1 + b_2) \sqrt{\rho} K, \quad (22)$$

where

$$\zeta = \frac{a_0}{2} \log(\rho), \quad (23)$$

$$\psi_1 = a_1 \log(\xi_1) + a_2 \log(\xi_2), \quad (24)$$

$$\psi_2 = b_1 \log(w_1) + b_2 \log(w_2). \quad (25)$$

Here a_0, a_1, a_2, b_1 , and b_2 are arbitrary constants.

There is another function $\mu = (b_1 + b_2) \zeta - a_0 \psi_2$ satisfies similar equation as ψ_1

$$\nabla_g^2 \mu = -a_0 (b_1 + b_2) R, \quad (26)$$

Using Eq.(20) the functions μ and ψ_1 satisfy a similar type of equation

$$\nabla_g^2 \sigma = -\frac{c}{4} g^{\alpha\beta} \text{tr}[(\partial_\alpha g^{-1}) \partial_\beta g], \quad (27)$$

where $c = (a_1 + a_2)$ when $\sigma = \psi_1$ and $c = -a_0(b_1 + b_2)$ when $\sigma = \mu$.

It is easy to show that

$$\xi_1 = \frac{w_2}{\det g_0 \sqrt{\rho}}, \quad \xi_2 = \frac{w_1}{\det g_0 \sqrt{\rho}} \quad (28)$$

Hence ψ_1 will not be considered as an independent function. It is interesting and important to note that under the minimality condition the matrix g satisfies the following condition as well.

Proposition 5. Minimality of S_0 , $H = 0$, also implies a sigma model [2], [3] like equation for g , i.e.,

$$\partial_\alpha [g^{\alpha\beta} g^{-1} \partial_\beta g] = 0 \quad (29)$$

3 Four Dimensions

Let the metric of a four dimensional manifold M_4 be given by

$$ds^2 = e^{2\psi} g_{\alpha\beta} dx^\alpha dx^\beta + \epsilon_1 g_{\alpha\beta} dy^\alpha dy^\beta, \quad (30)$$

where ψ is a function of x^α and $\epsilon_1 = \pm 1$. Local coordinate of M_4 are denoted as $x^a = (x^\alpha, y^\alpha)$, $a = 1 - 4$

Proposition 6. The Ricci flat equations $R_{ab} = 0$ for the metric (30) are given in two sets. One set satisfied identically due to the Proposition 5 above and the second one is given by

$$\nabla_g^2 \psi = 0 \quad (31)$$

There are two independent functions satisfying the above Laplace equation , ϕ and μ . Using (26) we find that $\psi = e_0 \phi + e_1 \mu$ where e_0 and e_2 are arbitrary constants and $b_2 = -b_1$. Combining all these constants we find that

$$e^{2\psi} = e^{2e_0 \phi} w_1^{-2m_1} w_2^{-2m_2}, \quad (32)$$

where m_1 and m_2 are constants satisfying $m_1 + m_2 = 0$. Then the line element (30) becomes

$$ds^2 = \frac{e^{2e_0 \phi}}{w_1^{2m_1} w_2^{2m_2}} \frac{h_{\alpha\beta} dx^\alpha dx^\beta}{\sqrt{\rho}} + \frac{h_{\alpha\beta} dy^\alpha dy^\beta}{\sqrt{\rho}}, \quad (33)$$

where ϕ satisfies the minimality condition ($H = 0$) (9) which is explicitly given by

$$[k_2 + \epsilon(\phi_{,y})^2] \phi_{,xx} - 2[k_0 + \epsilon \phi_{,x} \phi_{,y}] \phi_{,xy} + [k_1 + \epsilon(\phi_{,x})^2] \phi_{,yy} = 0, \quad (34)$$

where we take $(g_0)_{11} = k_1$, $(g_0)_{01} = k_0$, $(g_0)_{22} = k_2$ and assume that $\det(g_0) = k_1 k_2 - k_0^2 \neq 0$. Hence the functions w_1 and w_2 are given explicitly as

$$w_1 = k_1 + \epsilon(\phi_{,x})^2, \quad w_2 = k_2 + \epsilon(\phi_{,y})^2. \quad (35)$$

The metric in (33) with $e_0 = 0, m_1 = m_2 = 0$ reduces to an instanton metric [4].

4 Higher Dimensions

Let M_{2+2n} be a $2 + 2n$ dimensional manifold with a metric

$$ds^2 = e^{2\Phi} g_{\alpha\beta} dx^\alpha dx^\beta + G_{AB} dy^A dy^B, \quad (36)$$

where the local coordinates of M_{2+2n} are given by $x^{\alpha+A} = (x^\alpha, y^A)$, $A = 1, 2, \dots, 2n$, Φ and G_{AB} are functions of x^α alone. The Einstein equations are given in the following following proposition

Proposition 7. The Ricci flat equations for the metric in (36) are given by

$$\partial_\alpha [g^{\alpha\beta} G^{-1} \partial_\beta G] = 0, \quad (37)$$

$$\nabla_g^2 \Phi = \frac{1}{8} g^{\alpha\beta} \text{tr}[(\partial_\alpha G^{-1}) \partial_\beta G] + \frac{R}{2}, \quad (38)$$

where G is $2n \times 2n$ matrix of G_{AB} and G^{-1} is its inverse.

Let us choose G as a block diagonal matrix and each block is the 2×2 matrix g . This means that the metric in (36) reduces to a special form

$$ds^2 = e^{2\Phi} g_{\alpha\beta} dx^\alpha dx^\beta + \epsilon_1 g_{\alpha\beta} dy_1^\alpha dy_1^\beta + \dots + \epsilon_n g_{\alpha\beta} dy_n^\alpha dy_n^\beta, \quad (39)$$

where the local coordinates of M_{2+2n} are given by $x^{\alpha+A} = (x^\alpha, y_1^\alpha, \dots, y_n^\alpha)$, $\epsilon_i = \pm 1$, $i = 1, 2, \dots, n$. Then we have the following theorem

Theorem. For every two dimensional minimal surface S_0 immersed in a three dimensional manifold M_3 there corresponds a $2N = 2 + 2n$ -dimensional Ricci flat (pseudo-) Riemannian geometry with the metric given in (39) with

$$e^{2\Phi} = e^{2\psi} w_1^{-2n_1} w_2^{-2n_2} \rho^{n_1+n_2}, \quad (40)$$

where ψ is given in (32), w_1 and w_2 are given in (35), n_1 and n_2 satisfy

$$n_1 + n_2 = \frac{n-1}{2}. \quad (41)$$

Proof of the theorem: Using proposition 7 for the metric (39) the Ricci flat equations reduce to the following equation

$$\nabla_g^2 \Phi = \frac{n-1}{8} g^{\alpha\beta} \text{tr}[(\partial_\alpha g^{-1}) \partial_\beta g] \quad (42)$$

Hence, using (27) and letting $a_0 b_1 = n_1$, $a_0 b_2 = n_2$ and $\Phi = \mu$ we find (40) with the condition (41). All metric functions ψ , w_1 , w_2 and $g_{\alpha\beta}$ are expressed explicitly in terms the function ϕ and its derivatives $\phi_{,x}$ and $\phi_{,y}$. This means that for each solution ϕ of (34) there exists a $2N$ -dimensional metric (39). This completes the proof of the theorem.

The dimension of the manifold is $2N = 4 + 4(n_1 + n_2)$. Here $n = 1$ corresponds to the four dimensional case. The signature of the geometry depends on the signature of S . If the signature of S is zero then the signature of M_{2N} is also zero. If the signature of S is 2 then the signature of M_{2N} is $2(1 + \epsilon_1 + \dots + \epsilon_n)$.

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